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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WINGS OF MINIMUM DRAG

By Y. L. Zhilin

Translation

"Krylya minimalnogo soprotivleniya." Prikladnaya matematika i mekhanika, vol. XXI, 1957, pp. 213-220.

TMX # 56293

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NASA Evaluator

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

The variational problem of minimum drag for a given lift is considered for an arbitrary fixed planform with supersonic edges, subject to the restrictions of linear theory. It is shown that the velocity potential on the trailing-edge characteristic surface satisfies Laplace's equation with mixed boundary conditions. Computations of the drag of minimum-drag wings of quadrilateral planform indicate that only slight drag reductions from plane-wing values are obtained for cases where the theory is felt to be most valid. Some consideration is also given the minimum drag problem for a body with a fixed-tip cross section.

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WINGS OF MINIMUM DRAG*

By Y. L. Zhilin

The variational problem is considered for a wing of arbitrary fixed planform for a given lift. The wing is assumed to have supersonic edges and to have a slight disturbing effect on the free stream. It is shown that the potential of the disturbed velocity of the wing of minimum drag on the trailing-edge characteristic surface satisfies the Laplace equation with mixed boundary conditions. This result was earlier obtained by M. N. Kogan (ref. 1) for a wing having a straight trailing edge perpendicular to the undisturbed flow. The computation of the drag of wings of minimum drag of quadrilateral planform has shown that the drag of these wings differs slightly from the drag of plane wings. There is considered also the problem of the minimum drag of a body having a fixed-tip cross section, and a lower limit is obtained for the drag of such a body.

1. Consider the supersonic gas flow around an arbitrary body. The forces acting on the body can be represented in the form of integrals over an arbitrary closed surface enclosing the body. If the body has a small disturbing effect on the free stream, then

$$\left.\begin{aligned} X &= \frac{\rho_\infty}{2}\iint_{S_1}\left[(\beta^2u^2 + v^2 + w^2)\cos(nx) - 2uv\cos(ny) - 2uw\cos(nz)\right]dS_1 \\ Y &= -U_\infty\rho_\infty\iint_{S_1}[v\cos(nx) - u\cos(ny)]dS_1 \qquad (\beta^2 = M_\infty^2 - 1) \\ Z &= -U_\infty\rho_\infty\iint_{S_1}[w\cos(nx) - u\cos(nz)]dS_1 \end{aligned}\right\} \quad (1.1)$$

*"Krylya minimalnogo soprotivleniya." Prikladnaya matematika i mekhanika, vol. XXI, 1957, pp. 213-220.

where X, Y, and Z are the components of the force acting on the body, U_∞ and ρ_∞ the velocity and density of the free stream, u, v, and w the components of the disturbed flow, M_∞ is the free-stream Mach number, and n the outer normal to the surface S_1.

Moreover, on the surface S_1 the equation of conservation of mass must be observed:

$$U_\infty \sum = \iint_{S_1} \left[-\beta^2 u \cos(nx) + v \cos(ny) + w \cos(nz)\right] dS_1 \qquad (1.2)$$

where $\sum$ is the difference of the projected areas of the trailing and leading cross sections of the body on the plane $x = \text{constant}$.

These formulas can be considerably simplified if for the surface S_1 there is taken the surface formed by the forward and aft characteristic surfaces of the body n_1 and n_2, shown in figure 1. This method had been previously employed in considering the variational problem by A. A. Nikolski (ref. 2) for bodies of revolution and by M. N. Kogan (ref. 1) for a wing with a straight trailing edge perpendicular to the undisturbed flow.

On the leading characteristic surface the disturbed velocity is equal to zero. Hence, the integration in formulas (1.1) and (1.2) is conducted only along the trailing characteristic surface.

Let the equation of this surface be

$$x = f(y,z), \qquad (f_y^2 + f_z^2 = \beta^2) \qquad (1.3)$$

In this case, the cosines of the angles entering formulas (1.1) and (1.2) are:

$$\text{cox}(nx) = \frac{1}{M_\infty}, \qquad \cos(ny) = -\frac{f_y}{M_\infty}, \qquad \cos(nz) = -\frac{f_z}{M_\infty}$$

whence $dS_1 = M_\infty\, dy\, dz$ and formulas (1.1) and (1.2) become:

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$$\left.\begin{aligned}
X &= \frac{\rho_\infty}{2}\iint\limits_{S_2}(\beta^2u^2 + v^2 + w^2 + 2uvf_y + 2uwf_z)\,dy\,dz \\
Y &= -U_\infty\rho_\infty\iint\limits_{S_2}(v + uf_y)\,dy\,dz, \qquad Z = -V_\infty\rho_\infty\iint\limits_{S_2}(w + uf_z)\,dy\,dz \\
U_\infty\sum &= -\iint\limits_{S_2}(wf_z + vf_y + \beta^2u)\,dy\,dz
\end{aligned}\right\} \quad (1.4)$$

where S_2 is the projection of the trailing characteristic surface on the plane x = constant. Now introduce the potential $\varphi(x,y,z)$ of the disturbed velocity:

$$\varphi_\infty = u, \qquad \varphi_y = v, \qquad \varphi_z = w$$

Denote by $\varphi_0(y,z)$ the value of the potential φ on the trailing characteristic surface; that is,

$$\varphi_0(y,z) = \varphi[f(y,z),y,z]$$

Evidently,

$$\varphi_{0,y} = v + uf_y, \qquad \varphi_{0,z} = w + uf_z$$

If these equations and formula (1.3) are used, the expressions (1.4) can be reduced to the form

$$\left.\begin{aligned}
X &= \frac{\rho_\infty}{2}\iint\limits_{S_2}(\varphi_{0,y}^2 + \varphi_{0,z}^2)\,dy\,dz, \qquad Z = -U_\infty\rho_\infty\iint\limits_{S_2}\varphi_{0,z}\,dy\,dz \\
Y &= -U_\infty\rho_\infty\iint\limits_{S_2}\varphi_{0,y}\,dy\,dz, \qquad U_\infty\sum = -\iint\limits_{S_2}(\varphi_{0,y}f_y + \varphi_{0,z}f_z)\,dy\,dz
\end{aligned}\right\} \quad (1.5)$$

As was to be expected according to reference 1, the right sides of the formulas obtained depend only on the value of the potential on the trailing characteristic surface and do not depend on the normal derivative of the potential. The obtained formulas for the forces acting on the body are convenient also in that the expressions under the integral signs do not depend on the geometric properties of the trailing characteristic surface so that in solving the variational problems simple equations are obtained for the potential φ_0.

Now proceed to consider the variational problem for a lifting wing of arbitrary planform with supersonic edges. Assume that the wing has no thickness and lies in the plane $y = 0$. In this case, the functions φ and φ_0 will be asymmetrical with respect to the plane $y = 0$. Hence, the equation of conservation of mass is automatically satisfied, and in the computation of X and Y the integration can be carried out only over the upper half of the region S_2 symmetrical with respect to the z-axis; that is,

$$X = \rho_\infty \iint_S (\varphi_{0,y}^2 + \varphi_{0,z}^2) dy\, dz, \qquad Y = -2U_\infty \rho_\infty \iint_S \varphi_{0,y}\, dy\, dz \tag{1.6}$$

where S is the upper half of the region S_2.

That distribution of the potential in the region S is sought for which, for a given value of the lift Y, the drag X attains a minimum value. This is equivalent to seeking the minimum of the integral

$$J = \iint_S (\varphi_{0,y}^2 + \varphi_{0,z}^2 - 2q\varphi_{0,y}) dy\, dz$$

where q is the Lagrange constant determined by the magnitude of the lift.

The potential φ_0 must become zero on the projection of the line of intersection of the forward and aft characteristic surfaces on the plane x = constant. On the remaining part of the boundary of the region S the values of the potential φ_0 are initially unknown, and here the natural boundary condition must be obtained.

It is not difficult to show that the Euler equation for the functional J becomes the Laplace equation:

$$\varphi_{0,y,y} + \varphi_{0,z,z} = 0 \tag{1.7}$$

while the natural boundary condition determines the normal derivative:

$$\frac{\partial \varphi_0}{\partial n} = q \quad \text{for } y = 0,\ -\frac{1}{2}\, l \leqslant z \leqslant \frac{1}{2}\, l \tag{1.8}$$

where l is the wing span (fig. 2).

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Thus, the potential of the disturbed velocity of the wing of minimum drag on the aft characteristic surface satisfied the equation of Laplace with mixed boundary conditions. (In the given case the mixed problem can be reduced to the problem of Dirichlet with the aid of the substitution $\varphi_0 = \psi - qy$.) Strictly speaking, it is necessary to show further that the velocity potential distribution φ_0 obtained as a result of solving the Laplace equation corresponds to the flow around some wing.[1] In any case, φ_0 satisfying conditions (1.7) and (1.8) gives a lower estimate for the drag of wings of minimum drag, since in formulating the variational problem with respect to the potential φ_0, additional conditions, besides a given lift force, were not imposed.

It will be shown now with the aid of an example of wings of quadrilateral planform that this estimate differs by an insignificant amount from the drag of plane wings. Hence, the shape of wings of minimum drag will not be considered.

It should be remarked that condition (1.7) for the potential of the disturbed velocity of the wing of minimum drag may be obtained on the basis of the work of Jones (ref. 3). Jones obtained the result that the combined flow around a wing of minimum drag possesses the following properties: the pressure on the surface of the wing is equal to zero and the down-wash is constant, that is, on the surface of the wing

$$\frac{\partial \varphi_k}{\partial x} = 0, \qquad \frac{\partial \varphi_k}{\partial y} = q = \text{const.}$$

where φ_k is the potential of the combined flow satisfying the wave equation

$$-\beta^2 \varphi_{k,x,x} + \varphi_{k,y,y} + \varphi_{k,z,z} = 0$$

It is not difficult to see that the Cauchy problem exists for the derivative $\partial\varphi_k/\partial x$ satisfying the wave equation for the conditions

$$\frac{\partial \varphi_k}{\partial x} = 0, \qquad \frac{\partial}{\partial y}\left[\frac{\partial \varphi_k}{\partial x}\right] = 0 \qquad \text{on the surface of the wing}$$

If the uniqueness of the solution of the Cauchy problem is assumed, then within the entire region bounded by the forward and aft characteristic surfaces of the wing,

$$\partial\varphi_k/\partial x = 0 \qquad \text{or} \qquad \varphi_{k,y,y} + \varphi_{k,z,z} = 0$$

[1]If the forward and aft characteristic surfaces of the wing have a continuous normal (in this case the wing has a convex planform), this is evident.

From this, equations (1.7) and (1.8) for φ_0 follow directly. Conditions (1.7) and (1.8) permit constructing the formula for the drag of minimum-drag wings. Integrating (1.6) by parts and making use of equations (1.7) and (1.8) give

$$X = q\rho_\infty \int_{-l/2}^{l/2} \varphi_0 \, dz, \qquad Y = 2U_\infty\rho_\infty \int_{-l/2}^{l/2} \varphi_0 \, dz \tag{1.9}$$

The nondimensional coordinates and nondimensional potential are introduced:

$$\overline{y} = \frac{y}{l/2}, \qquad \overline{z} = \frac{z}{l/2}, \qquad \overline{\varphi}_0 = \frac{\varphi_0}{ql/2} \tag{1.10}$$

The potential $\overline{\varphi}_0$, as is not difficult to show, now does not depend on the magnitude of the lift and is determined only by the form of the line of intersection of the forward and aft characteristic surfaces on the plane $x = \text{constant}$. In the new variables it follows from equations (1.9) that

$$\frac{Y^2}{X} = \frac{U_\infty^2\rho_\infty}{2} 2l^2 \int_{-1}^{1} \overline{\varphi}_0 \, d\overline{z} \qquad \text{or} \qquad \frac{C_y^2}{C_x} = 2\lambda \int_{-1}^{1} \overline{\varphi}_0 \, d\overline{z} \tag{1.11}$$

where

$$C_y = \frac{Y}{\frac{1}{2} U_\infty^2\rho_\infty S^0}, \qquad C_x = \frac{X}{\frac{1}{2} U_\infty^2\rho_\infty S^0}$$

S^0 is the area of the wing, and λ is the wing aspect ratio.

The last formula shows that the ratio C_y^2/C_x for wings of minimum drag, as well as for plane wings, does not depend on the magnitude of the lift. By making use of equation (1.7) there can be proven still another property of the disturbed velocity u on the forward and aft characteristic surfaces. Since

$$\varphi_{0,y} = v + uf_y, \qquad \varphi_{0,y,y} = v_y + 2u_y f_y + u_x f_y^2 + uf_{y,y}$$

equation (1.7) can be written as

$$w_z + v_y + \beta^2 u_x + 2u_z f_z + 2u_y f_y + u(f_{y,y} + f_{z,z}) = 0$$

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At the same time the wave equation $w_x + v_y - \beta^2 u_x = 0$ holds throughout space. Hence, on the trailing characteristic surface of the wing of minimum drag,

$$u_y f_y + u_z f_z + u_x \beta^2 + \frac{f_{y,y} + f_{z,z}}{2} u = 0$$

or

$$u_{0,y} f_y + u_{0,z} f_z + \frac{f_{y,y} + f_{z,z}}{2} u_0 = 0 \tag{1.12}$$

where

$$u_0(y,z) = u(x,y,z) \qquad \text{for} \qquad x = f(y,z)$$

By using equation (1.3) for f, it is not difficult to prove the following property of the principal radius of curvature R of the trailing characteristic surface:

$$R = \frac{M_\infty}{f_{y,y} + f_{z,z}}, \qquad R_y f_y + R_z f_z = M_\infty$$

Equation (1.12) can then be reduced to the form

$$f_y(u_0\sqrt{R})_y + f_z(u_0\sqrt{R})_z = 0$$

This equation shows that the families of lines f = constant and $u_0\sqrt{R}$ = constant are orthogonal. The family of lines orthogonal to the family f = constant is determined by the equation

$$\frac{dy}{f_y} = \frac{dz}{f_z}$$

which is the equation of the bicharacteristics. Thus, on the trailing characteristic surface of the wing of minimum drag along a characteristic ray,

$$u_0\sqrt{R} = \text{const.} \tag{1.13}$$

The constant entering this expression varies from one ray to the next, and it can be determined only after solving the Goursat problem.

On the plane portions of the trailing characteristic surface along a characteristic ray, formula (1.13) assumes the form: u_0 = constant.

Formula (1.13) holds also for the leading characteristic surface because it has been shown that, in the combined flow in the region bounded by the leading and trailing characteristic surfaces, the pressure is equal to zero.

2. Compute the drag of minimum-drag wings of quadrilateral planform. In the preceding section it has been shown that the finding of the potential of the disturbed velocity in the trailing characteristic surface of a wing of minimum drag reduces to the solution of the mixed boundary problem for the equation of Laplace. This problem has in principle been solved by M. V. Keldish and L. I. Sedov, but the solution of concrete problems presents great formal difficulties. For this reason, it is in general more convenient to solve this problem not by the analytical method but by an electrointegrator. These computations are facilitated by the fact that for determining the drag it is sufficient to know the distribution of φ_0 along the trailing edge of the wing.

In the case of wings of quadrilateral planform (fig. 3) the projection of the line of intersection of the leading and trailing characteristic surfaces on the plane $x = \text{constant}$ consists of pieces of a circle, parabola, and straight line and is entirely determined by the parameters $m_1 = \beta^{-1}\text{tg}\chi_1$ and $m_2 = \beta^{-1}\text{tg}\chi_2$.

Since in this case $\lambda = 4/(m_1 - m_2)\beta$, it follows from formula (1.11) that

$$\beta \frac{C_y^2}{C_x} = \frac{8}{m_1 - m_2} \int_{-1}^{1} \bar{\varphi}_0 \, d\bar{z} \tag{2.1}$$

The right side of this equation depends only on the parameters m_1 and m_2.

For $m_1 = -m_2 = 1$, that is, for a wing of rhomboid planform with sonic edges, the region S is a semicircle and the potential φ_0 is readily found analytically. In this case,

$$\bar{\varphi}_0 = -r \sin \theta + \frac{1+r^2}{\pi r} \sin \theta \text{ arc tg}\left(\frac{2r}{1-r^2} \sin \theta\right) + \frac{1-r^2}{2\pi r} \cos \theta \ln \frac{1+r^2+2r \cos \theta}{1+r^2-2r \cos \theta} \tag{2.2}$$

where $r^2 = \bar{y}^2 + \bar{z}^2$ and $\text{tg}\theta = \bar{y}/\bar{z}$.

At the trailing edge,

$$\bar{\varphi}_0 = \frac{1 - \bar{z}^2}{\pi \bar{z}} \ln \frac{1 + \bar{z}}{1 - \bar{z}}$$

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For this wing,

$$\beta \frac{C_y^2}{C_x} = \frac{8}{\pi}\left(\frac{\pi^2}{4} - 1\right)$$

For the plane wing of the same planform,

$$\beta \frac{C_y^2}{C_x} = \frac{32}{3\pi}$$

that is, in the given case the drag of the plane wing is 10 percent greater than the drag of the minimum-drag wing. For other values of the parameters m_1 and m_2, the computation of the potential was made with the aid of the EI-11 electrointegrator, constructed according to the principle of electrical modeling. A preliminary comparison of the exact solution (e.g., (2.2)) with the solution computed by the electrointegrator showed that the error in computing by the electrointegrator does not exceed 1 percent for $\beta C_y^2/C_x$.

Figures 4 and 5 show the results of the computation of the potential $\overline{\varphi}_0$ along the trailing edge of the wings of minimum drag.

The computations show that for a sonic leading edge the drag of minimum-drag wings may differ considerably (by 12-22 percent) from the drag of plane wings of the same planform. However, this fact requires careful check since the applicability of the linear theory raises some doubt in this case. In those cases, however, where the leading edge is essentially supersonic and the applicability of the linear theory does not raise any doubt, the gain in drag reduction that is obtained is insignificant (of the order of 1-5 percent). The computations show that not only the curvature of the wing surface but its planform itself have a small effect on the value of the magnitude $\beta C_y^2/C_x$. The values of this magnitude for wings of minimum drag are (in the parentheses is given the decrease in drag in percent in comparison with plane wings):

m_1	m_2							
	-1.0	-0.8	-0.6	-0.2	0	0.2	0.4	0.6
0.4					4.04 (1)			
.6			3.74 (1)	3.93 (1)	4.08 (2)	4.34 (3.5)		
.8		3.70 (3)	3.73 (2.5)	3.97 (2)	4.18 (4.5)	4.54 (4.5)	4.95 (7.5)	
1.0	3.75 (10)	3.66 (4)	3.80 (6)	4.18 (9)	4.50 (12.5)	4.94 (14)	5.62 (19)	6.6 (22)

3. Derive the equation for the potential of the disturbed velocity of a body of minimum drag having a fixed-tip cross section. Consider the problem of the minimum-drag wing having a fixed-tip cross section.[1] The drag experienced by the body in supersonic flow and the equation of conservation of mass will be considered in the form (1.5).

We shall seek that distribution of potential φ_0 in the region S for which the minimum drag is attained for fixed value of Σ. It is not difficult to show that this minimum is attained for φ_0 satisfying the following conditions (fig. 6):

$$\varphi_{0,y,y} + \varphi_{0,z,z} = -q(f_{y,y} + f_{z,z}) \qquad \text{within the region } S$$

On the projection l_1 of the tip section in the plane x = constant (q is a Lagrange multiplier)

$$\frac{\partial \varphi_0}{\partial n} = -q\,\frac{\partial f}{\partial n}$$

On the projection l_2 of the line of intersection of the leading and trailing characteristic surfaces on the plane x = constant:

$$\varphi_0 = 0$$

The following function is introduced:

$$\psi = f + \frac{\varphi_0}{q} \tag{3.1}$$

As a result, the mixed boundary problem for the Laplace equation is also obtained:

$$\left.\begin{array}{l} \psi_{y,y} + \psi_{x,x} = 0 \qquad \text{within the region } S \\ \partial\psi/\partial m = 0 \qquad \text{on } l_1 \qquad \psi = f \qquad \text{on } l_2 \end{array}\right\} \tag{3.2}$$

Here, as previously, since the shape of the extremal bodies is of no interest, only the drag acting on these bodies is considered. Integrating by parts the expressions (1.5) for X and for $U_\infty\Sigma$ and making use of formulas (3.1) and (3.2) give

$$X = \frac{\rho_\infty U_\infty}{2}\,q\Sigma, \qquad U_\infty\Sigma = q\beta^2 S - qJ \qquad \left(J = \int_{l_2} f\,\frac{\partial\psi}{\partial n}\,dl\right)$$

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[1]Furthermore, there is fixed either the body length, if the body does not extend beyond the limits of the Mach cone, or its planform if the body has supersonic edges.

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or

$$X = \frac{\rho_\infty U_\infty^2}{2} \frac{\Sigma^2}{\beta^2 S - J}$$

The integral J entering this equation is always positive and is determined only by the shape of the region S. Hence, the following estimate for the drag of the arbitrary body holds:

$$X \geqslant \frac{\rho_\infty U_\infty^2}{2} \frac{\Sigma^2}{\beta^2 S}$$

In the case where the line of intersection of the forward and aft characteristic surfaces lies in the plane x = constant, this estimate coincides with the drag of bodies of minimum drag:

$$X_{min} = \frac{\rho_\infty U_\infty^2}{2} \frac{\Sigma^2}{\beta^2 S}$$

In particular, the bodies of revolution considered in the work of A. A. Nikolski (ref. 2) refer to this case.

All that has been said in this section applies also to bodies having cylindrical ducts.

REFERENCES

1. Kogan, M. N.: On Bodies of Minimum Drag in the Supersonic Flow of a Gas. PMM, vol. XXI, 1957, pp. 207-212.

2. Nikolski, A. A.: On Ducted Bodies of Revolution Having a Minimum Internal Drag in a Supersonic Flow. Trudy TsAGI, 1950.

3. Jones, R.: The Minimum Drag of Thin Wings in Frictionless Flow. Jour. Aero. Sci., vol. 18, no. 2, 1951, pp. 78-81.

Translated by S. Reiss
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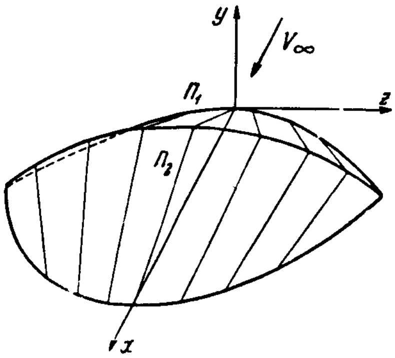

Figure 1.

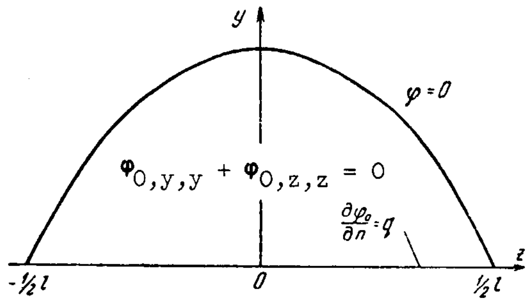

Figure 2.

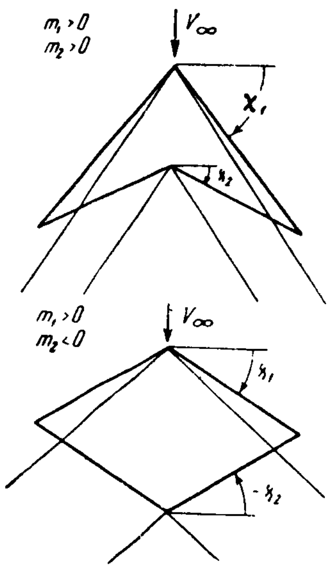

Figure 3.

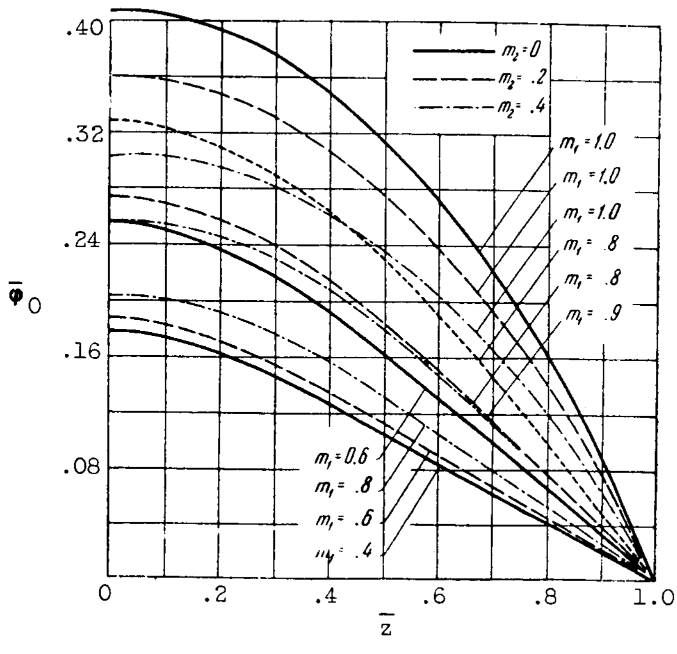

Figure 4.

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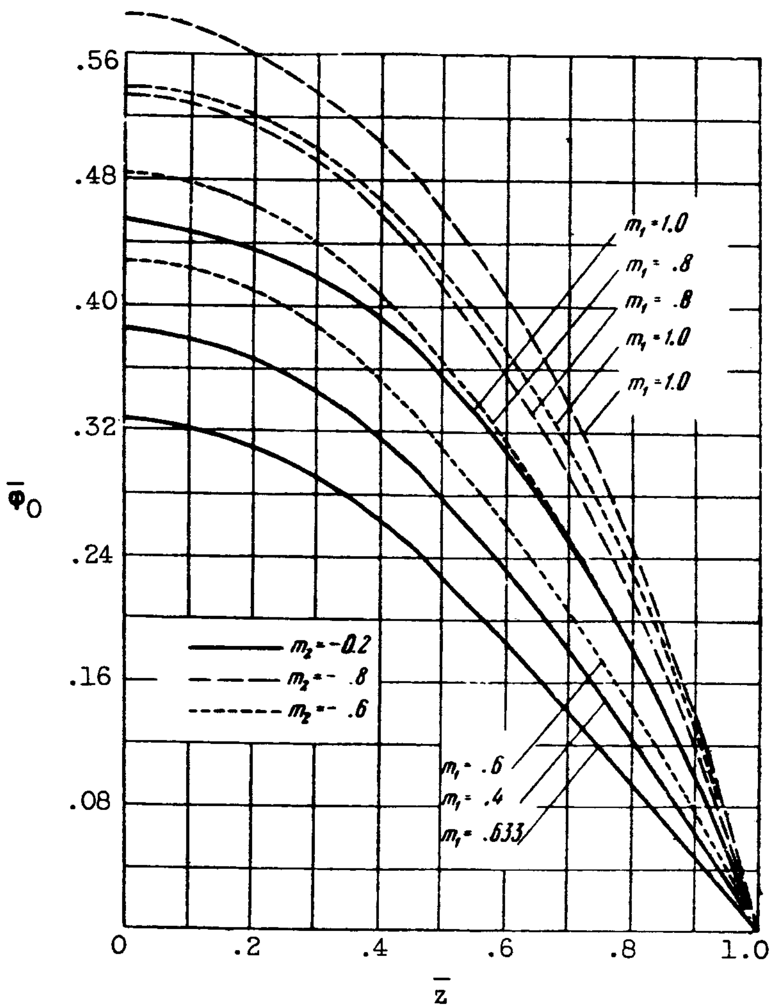

Figure 5.

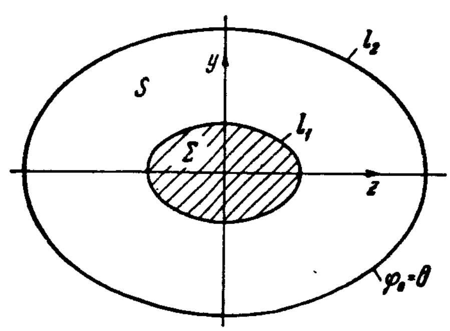

Figure 6.